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Normal frames and the validity of the equivalence principle: III. The case along smooth maps with separable points of self-intersection

Bozhidar Z Iliev†

Department Mathematical Modeling, Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, blvd. Tzarigradsko Chaussée 72, 1784 Sofia, Bulgaria

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Abstract. The equivalence principle is treated on a mathematically rigorous base on sufficiently general subsets of a differentiable manifold. This is carried out using the basis of derivations of tensor algebra over that manifold. Necessary and/or sufficient conditions of existence, uniqueness, and holonomicity of those bases, in which the components of the derivations of the tensor algebra over the manifold vanish on these subsets, are studied. The linear connections are considered in this context. It is shown that the equivalence principle is identically valid at any point, and along any path, in every gravitational theory based on linear connections. On higher-dimensional submanifolds it may be valid only in certain exceptional cases.

1. Introduction

In connection with the equivalence principle [1, ch 16], as well as from purely mathematical reasons [2–5], an important problem is the existence of local (holonomic or anholonomic [2]) coordinates (bases) in which the components of a linear connection [3] vanish on some subset, usually a submanifold, of a differentiable manifold [3]. This problem has been solved for torsion free, i.e. symmetric, linear connections [3, 4] in the cases at a point [2–5], along a smooth path without self-intersections [2, 5], and in a neighbourhood [2, 5]. These results were generalized in our previous works [6–9] for arbitrary derivations of the tensor algebra, with or without torsion, over a given differentiable manifold [3] and, in particular, for arbitrary linear connections. General results of this kind can be found in [10], where a criteria is presented for the existence of the above-mentioned special bases (coordinates) on submanifolds of a space with a symmetric affine connection.

This work is a revised version of [11] and a continuation of [7, 9]. It generalizes the results from [7, 9, 10] and deals with the problems of existence, uniqueness, and holonomicity of special bases (frames) in which the components of a derivation of the tensor algebra over a differentiable manifold vanish on some of its subsets of a sufficiently general type (sections 3 and 4). If such frames exist, they are called *normal*. In particular, the considered derivation may be a linear connection (section 5). In this context we make conclusions concerning the general validity and the mathematical formulation of the equivalence principle in a class of gravitational theories (section 6).

† E-mail address: bozho@inrne.acad.bg

2. Mathematical preliminaries

Below we reproduce for further reference purposes, as well as for the exact statement of the above problems, a few simple facts about derivations of tensor algebras that can be found in [7, 9] or derived from those in [3].

Let D be a derivation of the tensor algebra over a manifold M [12, 3]. By [3, proposition 3.3 of ch I] there exist a unique vector field X and a unique tensor field S of type $(1, 1)$ such that $D = L_X + S$. Here L_X is the Lie derivative along X [12, 3] and S is considered as a derivation of the tensor algebra over M [3].

If S maps from the set of C^1 vector fields into the tensor fields of type (1,1) and $S : X \mapsto S_X$, then the equation $D^S_X = L_X + S_X$ defines a derivation of the tensor algebra over M for any C^1 vector field X [3]. Such a derivation will be called an S-derivation along X and denoted for brevity simply by D_X. An S-derivation is a map D such that $D : X \mapsto D_X$, where D_X is an S-derivation along X.

Let $\{E_i, i = 1, \ldots, n := \dim(M)\}$ be a (coordinate or not [2, 4]) local basis (frame) of vector fields in the tangent bundle to M. It is holonomic (anholonomic) if the vectors $E_1, \ldots, E_n$ commute (do not commute) [2, 4]. Using the explicit action of L_X and S_X on tensor fields [3] one can easily deduce the explicit form of the local components of $D_X T$ for any C^1 tensor field T. In particular, the *components* $(W_X)^i_j$ of D_X are defined by

$$D_X(E_j) = (W_X)^i_j E_i. \tag{2.1}$$

Here and below all Latin indices, perhaps with some super- or subscripts, run from 1 to $n := \dim(M)$ and the usual summation rule on indices repeated on different levels is assumed. It is easily seen that $(W_X)^i_j := (S_X)^i_j - E_j(X^i) + C^i_{kj} X^k$ where $X(f)$ denotes the action of $X = X^k E_k$ on the C^1 scalar function f, as $X(f) := X^k E_k(f)$, and the C^i_{kj} define the commutators of the basic vector fields by $[E_j, E_k] = C^i_{jk} E_i$.

The change $\{E_i\} \mapsto \{E'_m := A^i_m E_i\}$, $A := [A^i_m]$ being a nondegenerate matrix function, implies the transformation of $(W_X)^i_j$ into (see (2.1)) $(W'_X)^m_l = (A^{-1})^m_i A^j_l (W_X)^i_j + (A^{-1})^m_i X(A^i_l)$. Introducing the matrices $W_X := [(W_X)^i_j]$ and $W'_X := [(W'_X)^m_l]$ and putting $X(A) := X^k E_k(A) = [X^k E_k(A^i_m)]$, we obtain

$$W'_X = A^{-1}\{W_X A + X(A)\}. \tag{2.2}$$

If ∇ is a linear connection with local components Γ^i_{jk} (see, e.g. [12, 3, 4]), then $\nabla_X(E_j) = (\Gamma^i_{jk} X^k) E_i$ [3]. Hence, we see from (2.1) that D_X is a covariant differentiation along X iff

$$(W_X)^i_j = \Gamma^i_{jk} X^k \tag{2.3}$$

for some functions Γ^i_{jk}.

Let D be an S-derivation and X and Y be vector fields. The *torsion operator* T^D of D is defined as

$$T^D(X, Y) := D_X Y - D_Y X - [X, Y]. \tag{2.4}$$

The S-derivation D is *torsion free* if $T^D = 0$ (cf [3]).

For a linear connection ∇, due to (2.3), we have $(T^\nabla(X, Y))^i = T^i_{kl} X^k Y^l$ where $T^i_{kl} := -(\Gamma^i_{kl} - \Gamma^i_{lk}) - C^i_{kl}$ are the components of the torsion tensor of ∇ [3].

Further we investigate the problem of the existence of bases $\{E'_i\}$ in which $W'_X = 0$ for an S-derivation D along any or a fixed vector field X. These bases (frames), if any, are called *normal*. Hence, due to (2.2), we have to solve the equation $W_X(A) + X(A) = 0$ with respect to A under conditions that will be presented below.

3. Derivations along every vector fields

This section is devoted to the existence and some properties of special bases (frames) $\{E'_i\}$, defined in a neighbourhood of a subset U of the manifold M, in which the components of an S-derivation D_X along an *every* vector field X vanish on U. These bases (frames), if any, are called *normal in* U.

The derivation D is called *linear on the set* $U \subseteq M$ if (cf (2.3)) in some (and hence in any) basis $\{E_i\}$ is fulfilled

$$W_X(x) = \Gamma_k(x) X^k(x) \tag{3.1}$$

where $x \in U$, $X = X^k E_k$, and Γ_k are some matrix functions on U. Evidently, a linear connection on M is a linear on U for every U (see (2.3)).

Proposition 3.1. If for some S-derivation D there exists a normal basis $\{E'_i\}$ in $U \subseteq M$, i.e. $W'_X|_U = 0$ for every vector field X, then D is linear on the set U.

Proof. Let us fix a basis $\{E_i\}$ and put $E'_i = A^j_i E_j$. Then $W'_X|_U = 0$, i.e. $W'_X(x) = 0$ for $x \in U$, which, in conformity with (2.2), is equivalent to (3.1) with $\Gamma_k = -(E_k(A))A^{-1}$, $A = [A^i_j]$. □

The opposite statement to proposition 3.1 is generally not true and for its appropriate formulation we need some preliminary results and explanations.

Let p be an integer, $p \geqslant 1$, and the Greek indices α and β run from 1 to p. Let J^p be a neighbourhood in $\mathbb{R}^p$ and $\{s^\alpha\} = \{s^1, \dots, s^p\}$ be (Cartesian) coordinates in $\mathbb{R}^p$.

Lemma 3.1. Let $Z_\alpha : J^p \to \mathrm{GL}(m, \mathbb{R})$, $\mathrm{GL}(m, \mathbb{R})$ being the group of $m \times m$ matrices on $\mathbb{R}$, be C^1 matrix-valued functions on J^p. Then the initial-value problem

$$\left.\frac{\partial Y}{\partial s^\alpha}\right|_s = Z_\alpha(s) Y \qquad Y|_{s=s_0} = \mathbb{1} \qquad \alpha = 1, \dots, p \tag{3.2}$$

where $\mathbb{1} := [\delta^i_j]^m_{i,j=1}$ is the unit matrix of the corresponding size, $s \in J^p$, $s_0 \in J^p$ is fixed, and Y is a $m \times m$ matrix function on J^p, has a solution, denoted by $Y = Y(s, s_0; Z_1, \dots, Z_p)$, which is unique and smoothly depends on all its arguments iff

$$R_{\alpha\beta}(Z_1, \dots, Z_p) := \frac{\partial Z_\alpha}{\partial s^\beta} - \frac{\partial Z_\beta}{\partial s^\alpha} + Z_\alpha Z_\beta - Z_\beta Z_\alpha = 0. \tag{3.3}$$

Proof. According to the results from [13, ch VI], in which $Z_1, \dots, Z_p$ are of class C^1, the integrability conditions for (3.2) are (cf [13, ch VI, equation (1.4)])

$$0 = \frac{\partial^2 Y}{\partial s^\alpha \partial s^\beta} - \frac{\partial^2 Y}{\partial s^\beta \partial s^\alpha} = \frac{\partial (Z_\beta Y)}{\partial s^\alpha} - \frac{\partial (Z_\alpha Y)}{\partial s^\beta}$$
$$= \frac{\partial Z_\beta}{\partial s^\alpha} Y - \frac{\partial Z_\alpha}{\partial s^\beta} Y + Z_\beta Z_\alpha Y - Z_\alpha Z_\beta Y = -R_{\alpha\beta}(Z_1, \dots, Z_p) Y.$$

Hence (see, e.g. [13, ch VI, theorem 6.1]) the initial-value problem (3.2) has a unique solution (of class C^2) iff (3.3) is satisfied. □

Let $p \leqslant n := \dim(M)$, $\alpha, \beta = 1, \dots, p$ and $\mu, \nu = p+1, \dots, n$. Let $\gamma : J^p \to M$ be a C^1 map. We suppose that for any $s \in J^p$ there exists its (p-dimensional) neighbourhood $J_s \subseteq J^p$ such that the restricted map $\gamma|_{J_s} : J_s \to M$ is without self-intersections, i.e. in J_s does not exist points s_1 and $s_2 \neq s_1$ with the property $\gamma(s_1) = \gamma(s_2)$. This assumption is equivalent to the one that the points of self-intersections of γ, if any, can be separated by neighbourhoods. With J_s^p we denote the union of all the neighbourhoods J_s with the above property; evidently, J_s^p is the maximal neighbourhood of s in which γ is without self-intersections.

Let us suppose at first that $J_s^p = J^p$, i.e. that γ is without self-intersection, and that $\gamma(J^p)$ is contained in a single-coordinate neighbourhood V of M.

Let us fix some one-to-one C^1 map $\eta : J^p \times J^{n-p} \to M$ such that $\eta(\cdot, \boldsymbol{t}_0) = \gamma$ for a fixed $\boldsymbol{t}_0 \in J^{n-p}$, i.e. $\eta(s, \boldsymbol{t}_0) = \gamma(s)$, $s \in J^p$. In $V \bigcap \eta(J^p, J^{n-p})$ we define coordinates $\{x^i\}$ by putting $(x^1(\eta(s, \boldsymbol{t})), \ldots, x^n(\eta(s, \boldsymbol{t}))) := (s, \boldsymbol{t}) \in \mathbb{R}^n$, $s \in J^p$, $\boldsymbol{t} \in J^{n-p}$.

Proposition 3.2. Let $\gamma : J^p \to M$ be a C^1 map without self-intersections and such that $\gamma(J^p)$ lies only in a one-coordinate neighbourhood. Let the derivation D be linear on $\gamma(J^p)$. Then a necessary and sufficient condition for the existence of a basis $\{E'_i\}$, defined in a neighbourhood of $\gamma(J^p)$, in which the components of D along every vector field vanish on $\gamma(J^p)$ is the validity in the above-defined coordinates $\{x^i\}$ of the equalities

$$[R_{\alpha\beta}(-\Gamma_1 \circ \gamma, \ldots, -\Gamma_p \circ \gamma)]|_{J^p} = 0 \qquad \alpha, \beta = 1, \ldots, p \tag{3.4}$$

where $R_{\alpha\beta}(\ldots)$ are defined by (3.3) for $m = n$ and $(s^1, \ldots, s^p) = s \in J^p$, i.e.

$$[R_{\alpha\beta}(-\Gamma_1 \circ \gamma, \ldots, -\Gamma_p \circ \gamma)](s) = \frac{\partial \Gamma_\alpha(\gamma(s))}{\partial s^\beta} - \frac{\partial \Gamma_\beta(\gamma(s))}{\partial s^\alpha} + (\Gamma_\alpha \Gamma_\beta - \Gamma_\beta \Gamma_\alpha)|_{\gamma(s)}. \tag{3.5}$$

Remark. This result was obtained by means of another method in [10] for the special case when D is a symmetric affine connection and U is a submanifold of M.

Proof. The following considerations will be made in the above-defined neighbourhood $V \bigcap \eta(J^p, J^{n-p})$ and coordinates $\{x^i\}$. Let $E_i = \partial/\partial x^i$.

Necessity. Let there exist a normal frame $\{E'_i = A_i^j E_i\}$ on $\gamma(J^p)$, i.e. $W'_X(\gamma(s)) = 0$, $s \in J^p$. By (2.2) the existence of $\{E'_i\}$ is equivalent to that of $A = [A_i^j]$, transforming $\{E_i\}$ into $\{E'_i\}$, and such that $[A^{-1}(W_X A + X(A))]|_{\gamma(s)} = 0$ for *every* X. As D is linear on $\gamma(J^p)$ (cf proposition 3.1), equation (3.1) is valid for $x \in \gamma(J^p)$ and some matrix-valued functions Γ_k. Consequently A must be a solution of $\Gamma'_k(x) = 0$, i.e. of

$$\Gamma_k(\gamma(s))A(\gamma(s)) + \left.\frac{\partial A}{\partial x^k}\right|_{\gamma(s)} = 0 \qquad s \in J^p. \tag{3.6}$$

Now define nondegenerate matrix-valued functions B and B_i by

$$A(\gamma(s)) = B(s) \qquad \left.\frac{\partial A}{\partial x^\alpha}\right|_{\gamma(s)} = \frac{\partial B(s)}{\partial s^\alpha} \qquad \alpha = 1, \ldots, p$$
$$\left.\frac{\partial A}{\partial x^\nu}\right|_{\gamma(s)} = B_\nu(s) \qquad \nu = p+1, \ldots, n.$$

Substituting these equalities into (3.6), we see that it splits into

$$\Gamma_\alpha(\gamma(s))B(s) + \frac{\partial B(s)}{\partial s^\alpha} = 0 \qquad \alpha = 1, \ldots, p \tag{3.7}$$
$$\Gamma_\nu(\gamma(s))B(s) + B_\nu(s) = 0 \qquad \nu = p+1, \ldots, n. \tag{3.8}$$

As these equations do not involve B_α, the B_α's are left arbitrary by (3.6), while the remaining B_i's are expressed via $B(s)$ through (see (3.8))

$$B_\nu(s) = -\Gamma_\nu(\gamma(s))B(s) \qquad \nu = p+1, \ldots, n. \tag{3.9}$$

So, $B(s)$ is the only quantity for determination. It must satisfy (3.7). If we arbitrarily fix the value $B(s_0) = B_0$ for a fixed $s_0 \in J^p$ and put $Y(s) = B(s)B_0^{-1}$ (B is nondegenerate as A is such by definition), we see that Y is a solution of the initial-value problem

$$\left.\frac{\partial Y}{\partial s^\alpha}\right|_s = -\Gamma_\alpha(\gamma(s))Y(s) \qquad \alpha = 1, \dots p \qquad Y|_{s=s_0} = \mathbb{1}_p = [\delta^i_j]^p_{i,j=1}. \tag{3.10}$$

By lemma 3.1 this initial-value problem has a unique solution $Y = Y(s, s_0; -\Gamma_1 \circ \gamma, \dots, -\Gamma_p \circ \gamma)$ iff the integrability conditions (3.4) are valid.

Consequently the existence of $\{E'_i\}$ (or of A) leads to (3.4).

Sufficiency. If (3.4) takes place, the general solution of (3.7) is

$$B(s) = Y(s, s_0; -\Gamma_1 \circ \gamma, \dots, -\Gamma_p \circ \gamma)B_0 \tag{3.11}$$

in which $s_0 \in J^p$ and the nondegenerate matrix B_0 are fixed. Consequently, admitting A to be a C^1 matrix-valued function, we see that in $V \bigcap \eta(J^p, J^{n-p})$ we can expand $A(\eta(s, \boldsymbol{t}))$, $s \in J^p$, $\boldsymbol{t} \in J^{n-p}$ up to second order terms with respect to $(\boldsymbol{t} - \boldsymbol{t}_0)$ as

$$\begin{aligned} A(\eta(s, \boldsymbol{t})) = {} & B(s) + B_i(s)[x^i(\eta(s, \boldsymbol{t})) - x^i(\eta(s, \boldsymbol{t}_0))] \\ & + B_{ij}(s, \boldsymbol{t}; \eta)[x^i(\eta(s, \boldsymbol{t})) - x^i(\eta(s, \boldsymbol{t}_0))][x^j(\eta(s, \boldsymbol{t})) - x^j(\eta(s, \boldsymbol{t}_0))] \end{aligned} \tag{3.12}$$

for the above-defined matrix-valued functions B, B_i, and some B_{ij}, which are such that $\det B(s) \neq 0, \infty$ and B_{ij} and their first derivatives are bounded when $\boldsymbol{t} \to \boldsymbol{t}_0$. (Note that in (3.12) the terms corresponding to $i, j = 1, \dots, p$ are equal to zero due to the definition of $\{x^i\}$.) In this case, due to (3.7)–(3.11), the general solution of (3.6) is

$$\begin{aligned} A(\eta(s, \boldsymbol{t})) = {} & \left\{\mathbb{1} - \sum_{\lambda=p+1}^{n} \Gamma_\lambda(\gamma(s))[x^\lambda(\eta(s, \boldsymbol{t})) - x^\lambda(\gamma(s))]\right\} \\ & \times Y(s, s_0; -\Gamma_1 \circ \gamma, \dots, -\Gamma_p \circ \gamma)B_0 \\ & + \sum_{\mu,\nu=p+1}^{n} \{B_{\mu\nu}(s, \boldsymbol{t}; \eta)[x^\mu(\eta(s, \boldsymbol{t})) - x^\mu(\gamma(s))][x^\nu(\eta(s, \boldsymbol{t})) - x^\nu(\gamma(s))]\} \end{aligned} \tag{3.13}$$

where $s_0 \in J^p$ and the nondegenerate matrix B_0 are fixed and $B_{\mu,\nu}$, $\mu, \nu = p + 1, \dots, n$, together with their first derivatives are bounded when $\boldsymbol{t} \to \boldsymbol{t}_0$. (The fact that only sums from $p + 1$ to n enter (3.13) is a consequence of $x^\alpha(\eta(s, \boldsymbol{t})) = x^\alpha(\gamma(s)) = s^\alpha$, i.e. $x^\alpha(\eta(s, \boldsymbol{t})) - x^\alpha(\eta(s, \boldsymbol{t}_0)) = x^\alpha(\eta(s, \boldsymbol{t})) - x^\alpha(\gamma(s)) = s^\alpha - s^\alpha \equiv 0$, $\alpha = 1, \dots, p$.)

Hence, from (3.4) follows the existence of a class of matrices $A(x)$, $x \in V \bigcap \eta(J^p, J^{n-p})$ such that the frames $\{E'_i = A^j_i E_j\}$ are normal for D (which is supposed to be linear on $\gamma(J^p)$). □

Thus bases $\{E'_i\}$ in which $W'_X = 0$ exist iff (3.4) is satisfied. If (3.4) is valid, then the normal bases $\{E'_i\}$ are obtained from $\{E_i = \partial/\partial x^i\}$ by means of linear transformations whose matrices must have the form (3.13).

Now we are ready to consider a general smooth (C^1) map $\gamma : J^p \to M$ whose points of self-intersection, if any, can be separated by neighbourhoods. For any $r \in J^p$ choose a coordinate neighbourhood $V_{\gamma(r)}$ of $\gamma(r)$ in M. Let there be given a fixed C^1 one-to-one map $\eta_r : J^p_r \times J^{n-p} \to M$ such that $\eta_r(\cdot, \boldsymbol{t}^r_0) = \gamma|_{J^p_r}$ for some $\boldsymbol{t}^r_0 \in J^{n-p}$. In the neighbourhood $V_{\gamma(r)} \bigcap \eta_r(J^p_r, J^{n-p})$ of $\gamma(J^p_r) \bigcap V_{\gamma(r)}$ we introduce local coordinates $\{x^i_r\}$ defined by

$$(x^1_r(\eta_r(s, \boldsymbol{t})), \dots, x^n_r(\eta_r(s, \boldsymbol{t}))) := (s, \boldsymbol{t}) \in \mathbb{R}^n$$

where $s \in J^p_r$ and $\boldsymbol{t} \in J^{n-p}$ are such that $\eta_r(s, \boldsymbol{t}) \in V_{\gamma(r)}$.

Theorem 3.1. Let the points of self-intersection of the C^1 map $\gamma : J^p \to M$, if any, be separable by neighbourhoods. Let the S-derivation D be linear on $\gamma(J^p)$, i.e. (3.1) to be valid for $x \in \gamma(J^p)$. Then a necessary and sufficient condition for the existence in some neighbourhood of $\gamma(J^p)$ of a basis $\{E'_i\}$ in which the components of D (along every vector field) vanish on $\gamma(J^p)$ is for every $r \in J$ in the above-defined local coordinates $\{x^i_r\}$ to be fulfilled

$$\left[R_{\alpha\beta}(-\Gamma_1 \circ \gamma, \ldots, -\Gamma_p \circ \gamma)\right](s) = 0 \qquad \alpha, \beta = 1, \ldots, p \tag{3.14}$$

where Γ_α are calculated by means of (3.1) in $\{x^i_r\}$, $R_{\alpha\beta}$ are given by (3.5), and $s \in J^p_r$ is such that $\gamma(s) \in V_{\gamma(r)}$.

Proof. For any $r \in J^p$ the restricted map $\gamma|_{'J^p_r} : {}'J^p_r \to M$, where ${}'J^p_r := \{s \in J^p_r, \gamma(s) \in V_{\gamma(s)}\}$, is without self-intersections (see the above definition of J^p_r) and $\gamma|_{'J^p_r}({}'J^p_r) = \gamma({}'J^p_r)$ lies in the coordinate neighbourhood $V_{\gamma(r)}$.

So, if a normal frame $\{E'_i\}$ exists for D, then, by proposition 3.2, equations (3.14) are identically satisfied.

Conversely, if (3.14) are valid, then, again, by proposition 3.2 for every $r \in J^p$ in a certain neighbourhood ${}'V_r$ of $\gamma({}'J^p_r)$ in $V_{\gamma(r)}$ exists a normal on $\gamma({}'J^p_r)$ basis $\{E^r_i\}$ for D_X along every vector field X. From the neighbourhoods ${}'V_r$ we can construct a neighbourhood V of $\gamma(J^p)$, for example by putting $V = \bigcup_{r \in J^p} {}'V_r$. Generally, V is sufficient to be taken as a union of ${}'V_r$ for some, but not all $r \in J^p$. On V we can obtain a normal basis $\{E'_i\}$ by putting $E'_i|_x = E^r_i|_x$ if x belongs to only one neighbourhood ${}'V_r$. If x belongs to more than one neighbourhood ${}'V_r$ we can choose $\{E'_i|_x\}$ to be the basis $\{E^r_i|_x\}$ for some arbitrary fixed r. □

Remark. Note that generally the basis obtained at the end of the proof of theorem 3.1 is not continuous in the regions containing intersections of several neighbourhoods ${}'V_r$. Hence it is, generally, no longer differentiable there. Therefore the adjective 'normal' is not very suitable in the mentioned regions. Maybe in such cases it is better to speak about 'special' frames instead of 'normal' ones.

Proposition 3.3. If on the set $U \subseteq M$ there exists normal frames on U for some S-derivation along every vector field, then all of them are connected by linear transformations whose coefficients are such that the action on them of the corresponding basic vectors vanishes on U.

Proof. If $\{E_i\}$ and $\{E'_i = A^j_i E_j\}$ are normal on U bases, i.e. if $W_X(x) = W'_X(x) = 0$ for $x \in U$ and every vector field $X = X^i E_i$, then due to (2.2), we have $X(A)|_U = 0$, i.e. $E_i(A)|_U = 0$. In contrast, if $W_X|_U = 0$ in $\{E_i\}$ and $E'_i = A^j_i E_j$ with $E_i(A)|_U = 0$, then from (2.2) follows $W'_X(x)|_U = 0$, i.e. $\{E'_i\}$ is also a normal basis. □

Proposition 3.4. If for some S-derivation D there exists a local holonomic normal basis on the set $U \subseteq M$ for D along every vector field, then D is torsion free on U. On the other hand, if D is torsion free on U and there exist smooth (C^1) normal bases on U for D along every vector field, then all of them are holonomic on U, i.e. their basic vectors commute on U.

Proof. If $\{E'_i\}$ is a normal basis on U, i.e. $W'_X(x) = 0$ for every X and $x \in U$, then using (2.1) and (2.4) (see also [6, equation (15)]), we find $T^D(E'_i, E'_j)|_U = -[E'_i, E'_j]|_U$. Consequently $\{E'_i\}$ is holonomic on U, i.e. $[E'_i, E'_j]|_U = 0$, iff $0 = T^D(X, Y)|_U = \{X'^i Y'^j T^D(E'_i, E'_j)\}|_U$ for every vector fields X and Y, which is equivalent to $T^D|_U = 0$.

Conversely, let $T^D|_U = 0$. We want to prove that any basis $\{E'_i = A^j_i E_j\}$ in which $W'_X = 0$ is holonomic on U. The holonomicity on U means $0 = [E'_i, E'_j]|_U = \{-(A^{-1})^l_k[E'_j(A^k_i) - E'_i(A^k_j)]E'_l\}|_U$. However, (see proposition 3.1 and (3.1)) the existence of $\{E'_i\}$ is equivalent to $W_X|_U = (\Gamma_k X^k)|_U$ for some functions Γ_k and every X. These two facts, combined with (2.1) and (2.4), lead to $(\Gamma_k)^i_j = (\Gamma_j)^i_k$. Using this and $\{\Gamma_k A + \partial A/\partial x^k\}|_U = 0$ (see the proof of proposition 3.1), we find $E'_j(A^k_i)|_U = -\{A^l_j A^m_i (\Gamma_l)^k_m\}|_U = (E'_i(A^k_j))|_U$. Therefore $[E'_i, E'_j]|_U = 0$ (see above), i.e. $\{E'_i\}$ is holonomic on U. □

4. Derivations along a fixed vector field

In this section we briefly outline some results concerning normal frames for (S-)derivations along a *fixed vector field*.

A derivation D_X is linear on $U \subseteq M$ along a *fixed* vector field X if (3.1) holds for $x \in U$ and the given X. In this sense, evidently, *any derivation along a fixed vector field is linear on every set* and, consequently, on the whole manifold M. Namely this is the cause due to which the analogue of proposition 3.1 for such derivations, which is evidently true, is absolutely trivial and does not even need to be formulated.

The existence of normal frames in which the components of D_X, with a *fixed* X, vanish on some set $U \subseteq M$ significantly differs from the same problem for D_X with an *every* X (see section 3). In fact, if $\{E'_i = A^j_i E_j\}$, $\{E_i\}$ being a fixed basis on U, is a normal frame on U, i.e. $W'_X|_U = 0$, then, due to (2.2), its existence is equivalent to the one of $A := [A^j_i]$ for which $(W_X A + X(A))|_U = 0$ for the *given* X. As X is *fixed*, the values of A at two different points, say $x, y \in U$, are connected through the last equation if and only if x and y lie on one and the same integral curve of X, the part of which between x and y belongs entirely to U. Hence, if $\gamma : J \to M$, J is an $\mathbb{R}$-interval, is (a part of) an integral curve of X, i.e. at $\gamma(s), s \in J$ the tangent to γ vector field $\dot\gamma$ is $\dot\gamma(s) := X|_{\gamma(s)}$, then along γ the equation $(W_X A + X(A))|_U = 0$ reduces to $\mathrm{d}A/\mathrm{d}s|_{\gamma(s)} = \dot\gamma(A)|_s = (X(A))|_{\gamma(s)} = -W_X(\gamma(s))A(\gamma(s))$. Using lemma 3.1 for $p = 1$, we see that the general solution of this equation is

$$A(s;\gamma) = Y(s, s_0; -W_X \circ \gamma)B(\gamma) \tag{4.1}$$

where $s_0 \in J$ is fixed, $Y = Y(s, s_0; Z)$, Z being a C^1 matrix function of s, is the unique solution of the initial-value problem (see [13, ch IV, section 1])

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = ZY \qquad Y|_{s=s_0} = \mathbb{1} \tag{4.2}$$

and the nondegenerate matrix $B(\gamma)$ may depend only on γ, but not on s. (Note that (4.2) is a special case of (3.2) for $p = 1$ and by lemma 3.1 it always has a unique solution because $R_{11}(Z_1) \equiv 0$ due to (3.3) for $p = 1$.)

From the above considerations, the next propositions follow.

Proposition 4.1. There exist normal bases for any S-derivation along a fixed vector field on every set $U \subseteq M$.

Proposition 4.2. The normal on the set $U \subseteq M$ bases for some S-derivation along a fixed vector field X are connected by linear transformations whose matrices are such that the action of X on them vanishes on U.

Proof. If $\{E_i\}$ and $\{E'_i = A^j_i E_j\}$ are such that $W'_X|_U = W_X|_U = 0$, then, due to (2.2), we have $X(A)|_U = 0$. On the other hand, if $W_X|_U = 0$ and $X(A)|_U = 0$, then, by (2.2), is fulfilled $W'_X|_U = 0$, i.e. $\{E'_i\}$ is normal. □

5. Linear connections

The results of section 3 can directly be applied to the case of linear connections. As this is more or less trivial, we present below only three such consequences.

Corollary 5.1. Let the points of self-intersection of the C^1 map $\gamma : J^p \to M$, if any, be separable by neighbourhoods, ∇ be a linear connection on M with local components Γ^i_{jk} (in a basis $\{E_i\}$) and $\Gamma_k := [\Gamma^i_{jk}]^n_{i,j=1}$. Then in a neighbourhood of $\gamma(J^p)$ there exists a normal frame $\{E'_i\}$ on $\gamma(J^p)$ for ∇, i.e. $\Gamma'_k|_{\gamma(J^p)} = 0$, iff for every $r \in J^p$ in the coordinates $\{x^i_r\}$ (defined before theorem 3.1) is satisfied (3.14) in which Γ_α, $\alpha = 1, \ldots, p$ are part of the components of ∇ in $\{x^i_r\}$ and $s \in J^p$ is such that $\gamma(s) \in V_{\gamma(r)}$.

Proof. For linear connections (3.1) is valid for every X in any basis. So, if in a basis $\{E'_i\}$ is fulfilled $W'_X|_U = 0$ for $U \subseteq M$, we have in it $\Gamma'_k|_U = 0$ (see (2.2)) and vice versa, if in a basis $\{E'_i\}$ is valid $\Gamma'_k|_U = 0$, then $W_X|_U = 0$ for every X. Combining this fact with theorem 3.1, we get the required result. □

Corollary 5.2. If on the set $U \subseteq M$ there exist normal frames for some linear connection on U, then these frames are connected by linear transformations whose matrices are such that the action of the corresponding basic vectors on them vanishes on U.

Proof. The result follows from proposition 3.3 and the proof of corollary 5.1. □

Corollary 5.3. Let, for some linear connection on a neighbourhood of some set $U \subseteq M$, there exist locally smooth normal bases on U. Then one (and hence any) such basis is holonomic on U iff the connection is torsion free on U.

Proof. The statement follows from (3.1) (or (2.3)) and proposition 3.4. □

6. Conclusion. The equivalence principle

Mathematically theorem 3.1 is the main result of this work. From the view point of its physical application, it expresses a sufficiently general necessary and sufficient condition for the existence of the normal frames considered here for tensor derivations, that, in particular, can be linear connections. For instance, it covers the problem on arbitrary submanifolds. In this sense, its special cases are the results from [10] and our previous papers [7, 9].

Let $\gamma\colon J^p \to M$, with J^p being a neighbourhood in $\mathbb{R}^p$ for some integer $p \leqslant \dim M$, be a C^1 map. If $p = 0$ or $p = 1$, then the conditions (3.14) are identically satisfied, i.e. $R_{\alpha\beta} = 0$ (see (3.5)). Hence, in these two cases normal bases along γ always exist (respectively at a point or along a path), which was already established in [7, 6] (and independently in [14]) and in [9] respectively.

In the other limiting case, $p = n := \dim(M)$, it is easily seen that the quantities (3.5) are simply the matrices formed from the components of the corresponding curvature tensor [7, 3, 4] and that the set $\gamma(J^p)$ consists of one or more neighbourhoods in M. Consequently, now theorem 3.1 states that the normal frames investigated here exist iff the corresponding derivation is flat, i.e. if its curvature tensor is zero, a result already found in [7].

In the general case, when $2 \leqslant p < n$ (for $n \geqslant 3$), normal bases, even anholonomic, do not exist if (and only if) conditions (3.14) are not satisfied. Besides, in this case the quantities (3.5) cannot be considered as a 'curvature' of $\gamma(J^p)$. They are something like 'commutators' of covariant derivatives of a type ∇_F, where F is a tangent to $\gamma(J^p)$ vector field (i.e. $F|_x \in T|_x(\gamma(J^p))$ if $\gamma(J^p)$ is a submanifold of M), and which act on a tangent to M vector fields.

Let us also note that the normal frames on a set U are generally anholonomic. They may be holonomic only in the torsion free case when the derivation's torsion vanishes on U.

The results of this work, as well as those of [7, 9], are important in connection with the use of normal frames in gravitational theories [1, 15]. In particular, we know that there exist normal frames (at a point or along paths) in Riemann–Cartan spacetimes, a problem that was open until recently [15].

The above results outline the general bounds of validity and express the exact mathematical form of the equivalence principle. This principle requires [1] that the gravitational field strength, theoretically identified with the components of a linear connection, can locally be transformed to zero by a suitable choice of the local reference frame (basis), i.e. by it there have to exist local bases in which the corresponding connection's components vanish.

The above discussion, as well as the results from [7, 9], show the identical validity of the equivalence principle in zero- and one-dimensional cases, i.e. for $p = 0$ and $p = 1$. Besides, these are the *only cases* when it is fulfilled for *arbitrary* gravitational fields. In fact, for $p \geqslant 2$ (in the case $n \geqslant 2$), as we saw in section 5, normal bases do not exist unless the conditions (3.14) are satisfied. In particular, for $p = n \geqslant 2$ it is valid only for flat linear connections (cf [7]).

Mathematically, the equivalence principle is expressed through corollary 5.1 or, in some more general situations, through theorem 3.1. Thus, we see that in gravitational theories based on linear connections this principle is identically satisfied at any fixed point or along any fixed path, but on submanifolds of dimensions greater than or equal to two it is generally not valid. Therefore in this class of gravitational theories the equivalence principle is a theorem derived from their mathematical background. It may play a role as a principle if one tries to construct a gravitational theory based on more general derivations, but then, generally, it will reduce such a theory to one based on linear connections.

A comprehensive analysis of the equivalence principle on the base of the present work and [7, 9] can be found in [16].

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